JOURNAL OF APPROXIMATION THEORY **40**, 148–154 (1984)

Lacunary Differentiability of Functions in $\mathbb{R}^n$

CALIXTO P. CALDERON

Department of Mathematics, University of Illinois, Chicago, Illinois 60680, USA

Communicated by Antoni Zygmund

Received March 31, 1983

The existence of directional derivatives of functions in the Sobolev spaces $L_k^p(\mathbb{R}^n)$ is studied. The novelty consists in calculating them through lacunary incremental quotients. Under these conditions no restrictions on p are necessary; the condition $p > (n/k)$ can be dropped.

0. INTRODUCTION

Several authors have studied the existence of differentials in the Stolz sense (total differentials) of functions in the Sobolev spaces $L_k^p(\mathbb{R}^n)$. (Here we use the notation of [7].) There are well-known classical results in this direction. Namely: If $f \in L_k^p(\mathbb{R}^n)$ and $p > (n/k)$, $1 \leqslant k \leqslant n$, then f possesses a total differential of order k almost everywhere in $\mathbb{R}^n$. In other words, if $\Delta f(x)$ denotes $f(x+h) - f(x)$ and $\Delta_n^k f(x) = \Delta_h(\Delta_h^{k-1} f(x))$, then

$$\lim_{|h| \to 0} \frac{|\Delta_h^k f(x) - \sum_{|\alpha| = k} h^\alpha D^\alpha f(x)|}{|h|^k} = 0$$

a.e. in $\mathbb{R}^n$.

If $p < n/k$, then one can construct a function in $L_k^p(\mathbb{R}^n)$ that is discontinuous every where. See [2, 5, 7, 9].

The last result was considerably refined in papers [1] and [3].

Paper [1] characterizes the Sobolev–Orlicz classes of functions having the property that all their functions possess a total differential of order 1 almost everywhere in $\mathbb{R}^n$.

Paper [3] extends this to the case $k > 1$. More precisely, we have: Let $\Psi(t)$ be non-negative, continuous, convex and increasing in $t \geqslant 0$. Consider the class Loc $L_k^\Psi(\mathbb{R}^n)$ of functions in $\mathbb{R}^n$ whose distributional derivatives $D^\alpha f$ satisfy

$$\int_{\mathrm{Loc}} \Psi(|D^\alpha f|)\, dx < \infty, \qquad |\alpha| \leqslant k, \tag{0.1}$$

where $1 \leqslant k < n$.

0021-9045/84 $3.00

If

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{k/(n-k)} dt < \infty, \tag{0.2}$$

then the functions of Loc $L_k^\Psi(\mathbb{R}^n)$ possess a total differential of order k almost every where in $\mathbb{R}^n$.

Conversely, if the integral (0.2) is divergent, then there is a function in Loc $L_k^\Psi(\mathbb{R}^n)$ that is discontinuous everywhere.

An equivalent characterization in terms of Lorentz spaces is discussed briefly in [8]. This characterization is an easy consequence of the relation[1]

$$\operatorname{loc} L_{n/k,1} = \bigcup_\Psi \operatorname{loc} L^\Psi(\mathbb{R}^n)$$

where

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{k/(n-k)} dt < \infty.$$

Here $L_{n/k,1}$ stands for the usual Lorentz space with parameters n/k and 1.

The aim of this paper is to study the existence of directional derivatives of functions in $L_k^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant n/k$, $1 \leqslant k \leqslant n$.

We shall prove the existence a.e. of directional derivatives when the incremental quotient is evaluated through lacunary increments. One of the typical results is that if $f \in L_1^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant n$, and β is a fixed vector in the unit sphere of $\mathbb{R}^n$, then the limit

$$\lim_{k\to\infty} 2^k\{f(x+2^{-k}\beta) - f(x)\} \tag{0.3}$$

exists a.e. in $\mathbb{R}^n$. Here k runs through the natural numbers.

This result is surprising considering the fact that f can be chosen to be discontinuous everywhere.

When calculating partial or directional derivatives through lacunary incremental quotients, the restriction on p ($p > n/k$) becomes unnecessary.

All the usual properties valid for functions in $C^k(\mathbb{R}^n)$ are recovered if one thinks in a.e. terms.

We shall discuss the main results in the next section.

[1] See Appendix.

1. Statement and Proof of the Main Results

We define $\Delta_{\beta,l} f(x) = \Delta_{\beta,l}^{(l)} f(x) = f(x + 2^{-l}\beta) - f(x)$, where β is a fixed vector in the unit sphere of $\mathbb{R}^n$ and l runs through the natural numbers. Likewise, we define

$$\Delta_{\beta,l}^{(m)} f(x) = \Delta_{\beta,l}(\Delta_{\beta,l}^{(m-1)} f(x)). \tag{1.1}$$

We introduce the maximal dyadic derivative

$$f_\beta^{*(m)}(x) = \sup_l 2^{lm} |\Delta_{\beta,l}^{(m)} f(x)|. \tag{1.2}$$

The dyadic directional derivative of order m of f at x in the direction of β is

$$\lim_{l\to\infty} 2^{lm} \Delta_{\beta,l}^{(m)} f(x) = D_\beta^m f(x). \tag{1.3}$$

Theorem A. *Let $f \in L_k^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$. Then*:

(i) $\|f_\beta^{*(k)}\|_p \leqslant C_p \|f\|_{p,k}$, $p > 1$.

(ii) meas $\{x : f_\beta^{*(k)}(x) > \lambda\} < C_1/\lambda \, \|f\|_{1,k}$ $\forall \lambda > 0$.

(iii) *If $1 \leqslant p \leqslant \infty$, then $D_\beta^k f$ exists a.e.*

Here $\|f\|_{p,k} = \sum_{0 \leqslant |\alpha| \leqslant k} \|D^\alpha f\|_p$, where $D^\alpha f$ denotes, as usual, a distributional derivative.

Proof. Following standard arguments it is enough to prove ((i) and (ii). Without loss of generality we may assume that $f \in C_0^\infty(\mathbb{R}^n)$ and consider the Taylor expansion of f about the point z:

$$\begin{aligned} f(s) = &\sum_{0 \leqslant |\alpha| \leqslant k-1} \frac{D^\alpha f(z)}{\alpha!} (s-z)^\alpha \\ &+ k \sum_{|\alpha| = k} (s-z)^\alpha \int_0^1 t^{k-1} D^\alpha f(s + t(z-s))\, dt. \end{aligned} \tag{1.4}$$

Consider now $\Delta_{\beta,l}^{(k)} f(x)$ and notice that

$$\Delta_{\beta,l}^{(k)} \left\{ \sum_{|\alpha| \leqslant k-1} \frac{D^\alpha f(z)}{\alpha!} (x-z)^\alpha \right\} = 0. \tag{1.5}$$

Set $x_0 = x$, $x_j = x + j2^{-l}\beta$, $1 \leqslant j \leqslant k$. Then:

$$|\Delta_{\beta,l}^{(k)} f(x)| \leqslant C \sum_{j=0}^{k} \sum_{|\alpha| = k} \int_0^{|z-x_j|} \rho^{k-1} \left| D^\alpha f\left(x_j + \rho \frac{z - x_j}{|z - x_j|}\right) d\rho \right|. \tag{1.6}$$

Integrating with respect to z over $|z-x|<3n2^{-l}$, we obtain

$$|\Delta_{\beta,l}^{(k)} f(x)| \leqslant C2^{nl} \sum_{k=0}^{k} \sum_{|\alpha|=k} \int_{|x-z|<3n2^{-l}} |x-z|^{-n+k} |D^\alpha f(x_j - z)| \, dz. \quad (1.7)$$

The above constant C depends on the dimension only. Let $K(x)$ be $|x|^{k-n}$ if $|x| \leqslant 6n$ and zero otherwise. Let us introduce the auxiliary kernels $K_j(x) = K(x - j\beta)$, $j = 0, 1, 2, \ldots, k$.

From the very definition we see that the kernels $K_j(x)$ are monotone functions of the distance from x to the fixed points $j\beta$, respectively.

Also the kernels $K_j(x)$ belong to $L \log^+ L$ over their support.

An application of Lemmas 1.3 and 1.4 in [4] completes the proof on account of the estimate

$$|2^{kl} \Delta_{\beta,l}^{(k)} f(x)| \leqslant C \sum_{|\alpha|=k} \sum_{j=0} \int 2^{nl} K_j(2^l(x-y)) |D^\alpha f(y)| \, dy. \quad (1.8)$$

2. Related Results

The existence of mixed derivatives is an easy consequence of Theorem A provided the derivatives are calculated through lacunary increments. Namely:

$$D^2_{\beta_1\beta_2} = D^1_{\beta_2}(D^1_{\beta_1} f).$$

In the case of Bessel Potential Spaces $\mathscr{L}^p_\alpha(\mathbb{R}^n)$ (using the notation of [7] we define

$$\begin{aligned} f_\beta^{*(\alpha)}(x) &= \sup_l 2^{\alpha l} |\Delta_{\beta,l}^{(k)} f(x)|, \\ f^{**(\alpha)}(x) &= \sup_\beta \sup_l 2^{\alpha l} |\Delta_{\beta,l}^{(k)} f(x)|, \end{aligned} \quad (2.1)$$

where k is the least integer $\geqslant \alpha$; $1 \leqslant \alpha < n$. We have in this case the inequality

$$\|f_\beta^{*(\alpha)}\|_p < C_p \|f\|_{p,\alpha}, \qquad 1 < p < \infty. \quad (2.2)$$

If $n-1 < \alpha < n$, we have

$$\|f^{**(\alpha)}\|_p < C_p \|f\|_{p,\alpha}, \qquad 1 < p < \infty. \quad (2.3)$$

$\|f\|_{p,\alpha}$ denotes the Bessel Potential norm introduced in [7].

The proofs of (2.2) and (2.3) are an easy consequence of the representation of Bessel Potentials, using minor modifications of Lemmas 1.3 and 1.4 of [4].

APPENDIX

Proof of

$$\operatorname{loc} L_{n/k,1} = \bigcup_{\Psi} \operatorname{loc} L^{\Psi}(\mathbb{R}^n), \tag{A1}$$

where

$$\int_1^{\infty} \left[\frac{t}{\Psi(t)}\right]^{k/(n-k)} dt < \infty.$$

An equivalent result is

THEOREM. *Let $g(r)$ be positive, decreasing and continuous in $(0, \infty)$. If $0 < \alpha < n$, then*

$$\int_0^1 g(r)\, r^{\alpha-n} r^{n-1}\, dr < \infty \tag{A2}$$

if and only if there exists a convex $\Psi(t) \geqslant 0$, $t > 0$, such that

$$\int_0^1 \Psi(g)\, r^{n-1}\, dr < \infty, \tag{A3}$$

$$\int_1^{\infty} \left[\frac{t}{\Psi(t)}\right]^{\alpha/(n-\alpha)} dt < \infty.$$

Proof. Let $g(r)$ be continuous for $r > 0$, decreasing and such that

$$\int_0^1 g(r)\, r^{\alpha-n} r^{n-1}\, dr < \infty. \tag{A4}$$

Let $\theta(s)$ be a positive convex function such that

$$\theta(r^{\alpha-n}) \sim g(r)\, r^{\alpha-n} \qquad \text{for} \quad r \to 0^+ \tag{A5}$$

(namely, $g(r)\, r^{\alpha-n}/\theta(r^{\alpha-n})$ and its reciprocal are bounded as $r \to 0^+$).

Clearly we have

$$\int_0^1 \theta(r^{\alpha-n})\, r^{n-1}\, dr < \infty. \tag{A6}$$

If Ψ is the conjugate of θ in the Orlicz sense, then

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{\alpha/(n-\alpha)} dt < \infty \tag{A7}$$

(for details see [3, Lemma e, p. 288]).

We have constructed θ to satisfy

$$\theta(r^{\alpha-n})/r^{\alpha-n} \sim g(r), \qquad r \to 0^+; \tag{A8}$$

but $\theta(s)\, s^{-1} \sim \theta'(s)$, which gives

$$\Psi'(g(r)) \sim r^{\alpha-n}, \qquad r \to 0^+ \tag{A9}$$

(because θ' and Ψ' are inverses of each other). On the one hand we have

$$\Psi'(g(r))\, g(r) \sim r^{\alpha-n} g(r), \qquad r \to 0^+, \tag{A10}$$

while on the other hand

$$\Psi'(s)\, s \sim \Psi(s), \qquad s \to \infty. \tag{A11}$$

Thus

$$\int_0^1 g(r)\, r^{\alpha-n} r^{n-1}\, dr < \infty \tag{A12}$$

implies

$$\int_0^1 \Psi(g)\, r^{n-1}\, dr < \infty.$$

This proves one direction. The other follows from Lemma d in [3, p. 289].

For further details the reader is advised to consult [3].

References

1. A. P. Calderon, On the differentiability of absolutely continuous functions, *Riv. Mat. Univ. Parma* **2** (1951), 203–214.
2. C. P. Calderon, E. B. Fabes, and N. M. Rivière, Maximal smoothing operators, *Indiana Univ. Math. J.* **23** (10) (1974), 889–898.
3. C. P. Calderon and J. Lewis, Maximal operators and some Orlicz classes, *Stud. Math.* **51** (1976), 285–296.
4. C. P. Calderon, Lacunary spherical means, III, *J. Math.* **23** (1979), 476–486.

5. JU. G. REŠETNJAK, Generalized derivatives and differentiability almost everywhere, *Soviet Math. Dokl.* **7** (5) (1966), 1273–1275.
6. J. SERRIN, On the differentiability of functions of several variables, *Arch. Rational Mech. Anal.* **7** (1961), 359–372.
7. E. M. STEIN, "Singular Integrals and Diffeentiability Properties of Functions in $\mathbb{R}^n$," Princeton Univ. Press, Princeton, N. J., 1970.
8. E. M. STEIN, Editor's note: The differentiability of functions in $\mathbb{R}^n$, *Ann. of Math.* **113** (1981), 383–385.
9. W. STEPANOV, Sur les conditions de l'existence de la differentielle totale, *Rec. Math. Soc. Moscow* **32** (1925), 511–526.